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Symmetrized nth powers of induced representations

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Abstract. A constructive method is given for obtaining all the symmetrized nth powers of an induced representation. These are the representations in one-one correspondence with the irreducible representations of the symmetric group S_n and they are obtained here as induced representations. This problem has only previously been solved in the case $n = 2$ by Mackey, and Bradley and Davies. Our starting point is the partial decomposition of the nth Kronecker power of an induced representation on subspaces labelled by double coset representatives. An action of the symmetric group is defined on these subspaces causing aggregates of them to form larger subspaces which may be symmetrized separately. This symmetrization is achieved by inducing through an intermediate subgroup so that a basis for the space is obtained which explicitly contains all permutations by elements of S_n.

1. Introduction

In this paper, constructive proofs are given which enable the nth Kronecker power of an induced representation to be reduced in terms of the symmetric group of degree n, so that, within each symmetry class, the reduction is expressed as a sum of induced representations. The results obtained have immediate application in the theory of crystallographic space groups since irreducible space group representations have convenient and elegant expression as induced representations. In particular, there is a direct application, in the case $n = 3$, to the Landau theory of second order phase transitions in crystals. A full account of this theory may be found in Lyubarskii (1960, chap 8) and Landau and Lifshitz (1958, chap 14).

In order to state the problem more precisely, we begin by defining the Kronecker power of a representation, exhibiting the natural action of the symmetric group on the basis of the tensor product space and showing that this leads to a decomposition into symmetry classes. Let G be a group and let V be the carrier space for a finite-dimensional representation Δ of G with basis $\{\psi_1, \ldots, \psi_d\}$. Then, for all elements $g \in G$, we have

$$g\psi_i = \sum_{j=1}^{d} \psi_j \Delta(g)_{ji}. \tag{1.1}$$

Now form the vector space Ω spanned by the ordered n-tuples of functions $(\psi_{i_1}, \ldots, \psi_{i_n})$, $i_s = 1, \ldots, d$; $s = 1, \ldots, n$. This is a carrier space for the nth Kronecker power of Δ. The action of any element $g \in G$ on Ω is given by

$$g(\psi_{i_1}, \ldots, \psi_{i_n}) = \sum_{j_1, \ldots, j_n} (\psi_{j_1}, \ldots, \psi_{j_n}) \Delta(g)_{j_1 i_1} \ldots \Delta(g)_{j_n i_n}. \tag{1.2}$$

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Also, there is an action of the symmetric group S_n on the positions of the functions in the n-tuple, given by

$$\pi(\psi_{i_1}, \ldots, \psi_{i_n}) = (\psi_{i_{\pi^{-1}(1)}}, \ldots, \psi_{i_{\pi^{-1}(n)}}), \tag{1.3}$$

for all $\pi \in S_n$. Clearly the actions of G and S_n commute. It may be shown that Ω will split into a direct sum of subspaces Ω^ν, in one-to-one correspondence with the UIR's (unitary irreducible representations) of S_n, such that each Ω^ν is invariant under both G and S_n. Ω^ν has the important property that if (g, π) belongs to $G \times S_n$, then Ω^ν carries the representation $\langle\nu\rangle(g) \otimes [\nu](\pi)$ of $G \times S_n$, where $[\nu]$ is a UIR of S_n of dimension f_ν and $\langle\nu\rangle$ is some UR (unitary representation) of G of dimension d_ν. The space Ω^ν is called the symmetry class corresponding to the UIR $[\nu]$ of S_n. This follows from the similar result for the general linear group $GL(d, C)$ of which the matrix representations of G form a subgroup. A proof of this general result was originally given by Schur and a clear exposition is to be found in Boerner (1970, chap 5).

It is a well known result, due to Mackey (1951), that a partial decomposition of the Kronecker square of an induced representation (more generally the nth Kronecker power of an induced representation) may be obtained through a double coset decomposition, since such representations are equivalent to induced representations defined on subspaces labelled by the double coset representatives. By analysing the subspaces of this decomposition, we show explicitly how to effect a further decomposition into symmetry classes. Also, we give a constructive method for obtaining the induced representations carried by these symmetry classes. This generalizes work of Mackey (1953), and Bradley and Davies (1970) who solved the problem in the case $n = 2$.

We now introduce some notation and briefly develop the background theory in order to describe the method used. Let G be a finite group, K a subgroup of G and D a UR of K with orthonormal basis $\{\phi_1, \ldots, \phi_d\}$. The left coset decomposition of G relative to K is given by

$$G = \bigcup_{\sigma=1}^{h} q_\sigma^0 K \tag{1.4}$$

where $h = |G:K|$. The vector space, of dimension hd, spanned by the basis functions $\{q_\sigma^0 \phi_i : \sigma = 1, \ldots, h;\ i = 1, \ldots, d\}$ is a carrier space for the induced representation $D \uparrow G$ of G. This construction is described more fully by Bradley (1966). It can be shown that $D \uparrow G$ consists of block matrices labelled by the left coset representatives so that the (γ, τ) block is given by

$$(D \uparrow G)_{\gamma\tau}(g) = D[(q_\gamma^0)^{-1} g q_\tau^0]\, \delta_{\gamma, g\tau} \tag{1.5}$$

where $\delta_{\gamma,g\tau}$ is unity if $gq_\tau^0 \in q_\gamma^0 K$ and zero otherwise. Backhouse and Bradley (1970) have shown that this definition is identical to the one given by Mackey (1952) if K is of finite index in G.

The double coset decomposition of G relative to K is

$$G = \bigcup_j K d_{\alpha_j}^1 K. \tag{1.6}$$

For each $d_{\alpha_1} \in \{d_{\alpha_j}^1\}$ define the subgroup $K_1(\alpha) = K \cap d_{\alpha_1} K d_{\alpha_1}^{-1}$ and let the left coset decomposition of K relative to K_1 be given by

$$K = \bigcup_\sigma q_\sigma^1 K_1. \tag{1.7}$$

In general we define the double coset decomposition

$$G = \bigcup_j K_{r-1} d^r_{\alpha_j} K \qquad (r = 1, \ldots, n-1) \tag{1.8}$$

where $K_0 = K$ and

$$K_r = K \cap \bigcap_{i=1}^{r} d_{\alpha_i} K d_{\alpha_i}^{-1}, \qquad (r = 1, \ldots, n-1). \tag{1.9}$$

K_r depends on the fixed set $d_{\alpha_i} \in \{d^i_{\alpha_j}\}$, $i = 1, \ldots, r$, of double coset representatives chosen. Each set $\{d^i_{\alpha_j}\}$ is defined with respect to the group K_{i-1} of equation (1.9) and hence we obtain the chains of double coset representatives which are used to label the subspaces in the partial decomposition of Ω. Also we have a left coset decomposition

$$K_{r-1} = \bigcup_\sigma q^r_\sigma K_r \qquad (r = 1, \ldots, n-1). \tag{1.10}$$

Define the UR D_α of $K_\alpha = d_\alpha K d_\alpha^{-1}$ on the space spanned by the functions

$$\{d_\alpha \phi_i : i = 1, \ldots, d\}$$

as follows:

$$D_\alpha(d_\alpha k d_\alpha^{-1}) = D(k), \tag{1.11}$$

for all $k \in K$. We are now in a position to write down explicitly the decomposition of the nth Kronecker power of an induced representation as a direct sum of induced representations:

$$\begin{aligned}(D \uparrow G) &\otimes (D \uparrow G) \otimes \ldots n \text{ times} \\ &\equiv \bigoplus_{(\alpha)} [(D_{\alpha_{n-1}} \downarrow K_{n-1}) \otimes \ldots \otimes (D_{\alpha_1} \downarrow K_{n-1}) \otimes (D \downarrow K_{n-1})] \uparrow G,\end{aligned} \tag{1.12}$$

where the direct sum ranges over all chains of double coset representatives

$$(\alpha) = (d_{\alpha_{n-1}}, \ldots, d_{\alpha_1}, 1)$$

defined above and where the downward arrow denotes subduction. Bradley (1966) has proved the above theorem for finite groups. Mackey (1952) has proved this result, in the case $n = 2$, for a closed subgroup K of a locally compact group G. Hence the above result will still be true if we take K to be a closed subgroup of finite index in G and take D to be a finite-dimensional representation of G.

This partial decomposition of the nth Kronecker power of $(D \uparrow G)$ leads to a decomposition of the space Ω into a direct sum of G-invariant spaces $\Omega_{(\alpha)}$ defined by

$$\Omega_{(\alpha)} = \sum q^0_\sigma q^1_{\sigma'} \ldots q^{n-1}_{\sigma''}(d_{\alpha_{n-1}}\phi_i, \ldots, d_{\alpha_1}\phi_j, \phi_k) \tag{1.13}$$

where, for fixed (α), the sum is over all $\sigma, \sigma', \ldots; i, j, \ldots$. The summation sign means that we take the linear span of all the functions on the right-hand side of the equation. In § 2 we study the dependence of the space $\Omega_{(\alpha)}$ on the set of double coset representatives (α) and find conditions under which two distinct n-tuples of elements of G may be labels for the same space. The next step is to define an action of S_n on the spaces $\Omega_{(\alpha)}$ by permuting the order of the elements in the n-tuple (α). It is found that the $n!$ spaces obtained, which need not all be distinct, are all present in the direct sum decomposition of Ω and carry equivalent representations of the group G. The direct sum of the distinct spaces obtained, which we denote by $T(\alpha)$, is invariant under both G and S_n, and so we may restrict our attention to the decomposition of $T(\alpha)$ into its symmetrized parts. The special case when all the double coset representatives in the n-tuple (α) are distinct is considered in § 5 and the general case in § 6.

2. Double coset representatives

It was shown in the last section how a set of double coset representatives may be used to label the invariant subspaces of the direct sum decomposition. But, by inspection of equation (1.8), we see that any n-tuple $(\alpha) = (d_{\alpha_{n-1}}, \ldots, d_{\alpha_1}, 1)$ of elements of $\boldsymbol{G}$ could be the label of a subspace for some choice of double coset representatives. In this section we give the condition that two ordered n-tuples

$$(\alpha) = (d_{\alpha_{n-1}}, \ldots, d_{\alpha_1}, d_{\alpha_0}) \qquad \text{and} \qquad (\beta) = (d_{\beta_{n-1}}, \ldots, d_{\beta_1}, d_{\beta_0}),$$

with $d_{\alpha_0} = d_{\beta_0} = 1$, should be labels for the same subspace. However, in the construction we give for symmetrizing spaces, such a general prescription for choosing n-tuples is unsatisfactory. Later in this section, we give a procedure for obtaining the double coset representatives which lead to the maximum number of equal entries, and in § 4 we show the essential uniqueness of such a choice.

Considering the relations between alternative sets of double coset representatives leads us to the following definition:

$(\alpha) \sim (\beta)$ if

$$d_{\alpha_r} = p_{r-1} \ldots p_1 p_0 d_{\beta_r} k_{r-1} \qquad (r = 1, \ldots, n-1) \tag{2.1}$$

where $p_i \in \boldsymbol{K}_i^\alpha$, $k_i \in \boldsymbol{K}$ $(i = 0, \ldots, n-2)$. The upper index α on $\boldsymbol{K}_i^\alpha$ distinguishes the groups defined by equation (1.9) from the corresponding groups defined with respect to the n-tuple (β). In order to prove that $\sim$ is an equivalence relation, we need the following lemma.

Lemma (2.1).

$$\boldsymbol{K}_r^\alpha = p_{r-1} \ldots p_0 \boldsymbol{K}_r^\beta p_0^{-1} \ldots p_{r-1}^{-1}.$$

Proof.

The proof is by induction. From equation (1.9)

$$\begin{aligned} \boldsymbol{K}_{r+1}^\alpha &= \boldsymbol{K}_r^\alpha \cap d_{\alpha_{r+1}} \boldsymbol{K} d_{\alpha_{r+1}}^{-1} \\ &= \boldsymbol{K}_r^\alpha \cap p_r \ldots p_0 d_{\beta_{r+1}} \boldsymbol{K} d_{\beta_{r+1}}^{-1} p_0^{-1} \ldots p_r^{-1} \\ &= p_r \ldots p_0 \boldsymbol{K}_{r+1}^\beta p_0^{-1} \ldots p_r^{-1}. \end{aligned}$$

The second line follows by equation (2.1) and the third from the inductive hypothesis, since $p_r \in \boldsymbol{K}_r^\alpha$. Clearly the proof for $r = 1$ is identical if we take $\boldsymbol{K}_0^\alpha = \boldsymbol{K}$. Hence the result follows by induction.

Theorem (2.2).

The relation $\sim$ is an equivalence relation.

Proof.

(i) $(\alpha) \sim (\alpha)$ if we choose $p_i = k_i = 1$.

(ii) If $(\alpha) \sim (\beta)$, then we may invert equation (2.1) to obtain

$$\begin{aligned} d_{\beta_r} &= p_0^{-1} \dots p_{r-1}^{-1} d_{\alpha_r} k_{r-1}^{-1} \\ &= (p_0^{-1} p_1^{-1} \dots p_{r-1}^{-1} p_{r-2} \dots p_0) \dots \\ &\quad \dots (p_0^{-1} p_1^{-1} p_2^{-1} p_1 p_0)(p_0^{-1} p_1^{-1} p_0) p_0^{-1} d_{\alpha_r} k_{r-1}^{-1}, \end{aligned}$$

where $r = 1, \dots, n-1$. But by lemma (2.1)

$$p_0^{-1} \dots p_{r-1}^{-1} p_{r-2} \dots p_0 \in p_0^{-1} \dots p_{r-2}^{-1} \boldsymbol{K}_{r-1}^{\alpha} p_{r-2} \dots p_0 = \boldsymbol{K}_{r-1}^{\beta}.$$

Hence $(\beta) \sim (\alpha)$.

(iii) If $(\alpha) \sim (\beta)$ and $(\beta) \sim (\gamma)$ we also have the relation

$$d_{\beta_r} = s_{r-1} \dots s_0 d_{\gamma_r} h_{r-1} \qquad (r = 1, \dots, n-1) \tag{2.2}$$

where $s_i \in \boldsymbol{K}_i^{\beta}$, $h_i \in \boldsymbol{K}$, $(i = 0, \dots, n-2)$. Substituting equation (2.2) into equation (2.1), we obtain

$$\begin{aligned} d_{\alpha_r} &= p_{r-1} \dots p_0 s_{r-1} \dots s_0 d_{\gamma_r} h_{r-1} k_{r-1} \\ &= [p_{r-1}(p_{r-2} \dots p_0 s_{r-1} p_0^{-1} \dots p_{r-2}^{-1})] \dots \\ &\quad \dots [p_1(p_0 s_1 p_0^{-1})](p_0 s_0) d_{\gamma_r}(h_{r-1} k_{r-1}). \end{aligned}$$

But $p_i \in \boldsymbol{K}_i^{\alpha}$ and $p_{i-1} \dots p_0 s_i p_0^{-1} \dots p_{i-1}^{-1} \in \boldsymbol{K}_i^{\alpha}$ by lemma (2.1). Hence, $(\alpha) \sim (\gamma)$ and $\sim$ is an equivalence relation.

The next step is to show that there is a one-to-one correspondence between the equivalence classes of n-tuples and the subspaces $\Omega_{(\alpha)}$ defined by the direct sum decomposition of Ω.

Theorem (2.3).

Let (α) and (β) be any two n-tuples of elements of G with $d_{\alpha_0} = d_{\beta_0} = 1$. If $(\alpha) \sim (\beta)$ then $\Omega_{(\alpha)} = \Omega_{(\beta)}$.

Proof.

Let

$$G = \bigcup_{\sigma} w_{\sigma}^{0} \boldsymbol{K}, \qquad \boldsymbol{K}_{r-1}^{\beta} = \bigcup_{\sigma} w_{\sigma}^{r} \boldsymbol{K}_{r}^{\beta} \qquad (r = 1, \dots, n-1).$$

Conjugating the latter relation by $(p_{r-1} \dots p_0)$, $r \geqslant 1$, and using lemma (2.1), we obtain

$$\begin{aligned} \boldsymbol{K}_{r-1}^{\alpha} &= \bigcup_{\sigma} p_{r-1} \dots p_0 w_{\sigma}^{r} \boldsymbol{K}_{r}^{\beta} p_0^{-1} \dots p_{r-1}^{-1} \\ &= \bigcup_{\sigma} p_{r-1} \dots p_0 w_{\sigma}^{r} p_0^{-1} \dots p_{r-1}^{-1} \boldsymbol{K}_{r}^{\alpha}. \end{aligned} \tag{2.3}$$

Since $p_{r-1} \in \boldsymbol{K}_{r-1}^{\alpha}$, we have

$$\boldsymbol{K} = \bigcup_{\sigma} w_{\sigma}^{1} p_0^{-1} \boldsymbol{K}_{1}^{\alpha}$$

and for $r = 2, \dots, n-1$

$$\boldsymbol{K}_{r-1}^{\alpha} = \bigcup_{\sigma} p_{r-2} \dots p_0 w_{\sigma}^{r} p_0^{-1} \dots p_{r-1}^{-1} \boldsymbol{K}_{r}^{\alpha}.$$

Hence

$$G = \bigcup_{\sigma, \sigma', \dots, \sigma''} w_{\sigma}^{0} w_{\sigma'}^{1} \dots w_{\sigma''}^{n-1} p_0^{-1} \dots p_{n-2}^{-1} \boldsymbol{K}_{n-1}^{\alpha}.$$

From definition (1.13)

$$\Omega_{(\beta)} = \sum w^0_\sigma \dots w^{n-1}_{\sigma''}(d_{\beta_{n-1}}\phi_i, \dots, d_{\beta_1}\phi_j, \phi_k)$$
$$\Omega_{(\alpha)} = \sum w^0_\sigma \dots w^{n-1}_{\sigma''} p_0^{-1} \dots p_{n-2}^{-1}(d_{\alpha_{n-1}}\phi_i, \dots, d_{\alpha_1}\phi_j, \phi_k).$$

In order to reduce this expression to a simpler form, we use the fact that the carrier space for the UR D of $\boldsymbol{K}$, with basis $\{\phi_1, \dots, \phi_d\}$ is invariant under the action of $\boldsymbol{K}$. Hence the functions $\{p_0^{-1} \dots p_{n-2}^{-1}\phi_k : k = 1, \dots, d\}$ span the same space as the functions $\{\phi_1, \dots, \phi_d\}$. Also, if $t \geqslant r$, $p_t \in \boldsymbol{K}_r^\alpha \subset d_{\alpha_r}\boldsymbol{K}d_{\alpha_r}^{-1}$, so

$$p_0^{-1} \dots p_{n-2}^{-1} d_{\alpha_r}\phi_i = p_0^{-1} \dots p_{r-1}^{-1} d_{\alpha_r} k\phi_i$$
$$= d_{\beta_r} k_{r-1} k\phi_i,$$

where $k \in \boldsymbol{K}$ and the last line follows from equation (2.1). Applying the above argument, it can be seen that the functions $\{p_0^{-1} \dots p_{n-2}^{-1} d_{\alpha_r}\phi_i : i = 1, \dots, d\}$ span the same space as the functions $\{d_{\beta_r}\phi_i : i = 1, \dots, d\}$. Hence $\Omega_{(\alpha)} = \Omega_{(\beta)}$.

Thus we have the result that the decomposition of the space Ω as a direct sum of subspaces $\Omega_{(\alpha)}$ is independent of the particular choice of double coset representative at each stage. Since we wish to symmetrize these spaces, we now give a prescription for choosing the double coset representatives in a manner that ensures the maximum number of equal entries in each indexing n-tuple (α). We prove this statement in § 4 and show that our choice leads to an unambiguous and essentially unique way of labelling the subspaces of Ω.

Define sets A_i as follows:

$$A_0 = \{1\}$$
$$A_1 = \{d_{\alpha_1} : \boldsymbol{G} = \bigcup \boldsymbol{K} d_{\alpha_1} \boldsymbol{K} \text{ and } A_0 \subset A_1\}.$$

For each $d_{\alpha_1} \in A_1$, define

$$A_2(\alpha_1) = \{d_{\alpha_2} : \boldsymbol{G} = \bigcup \boldsymbol{K}_1(\alpha_1) d_{\alpha_2} \boldsymbol{K},\ A_1 \subset A_2(\alpha_1)\}.$$

By $A_1 \subset A_2(\alpha_1)$ we mean that distinct members of A_1 are distinct members of $A_2(\alpha_1)$. This is possible because $\boldsymbol{K}_1(\alpha_1) \subset \boldsymbol{K}$. Likewise for $r = 3, \dots, n-1$ we define

$$\begin{aligned} A_r(\alpha_1, \dots, \alpha_{r-1}) &= \{d_{\alpha_r} : \boldsymbol{G} = \bigcup \boldsymbol{K}_{r-1}(\alpha_1, \dots, \alpha_{r-1}) d_{\alpha_r} \boldsymbol{K} \text{ with } A_{r-1}(\alpha_1, \dots, \alpha_{r-2}) \\ &\qquad \subset A_r(\alpha_1, \dots, \alpha_{r-1})\}. \end{aligned}$$

Hence we obtain sequences of sets of the form

$$A_{n-1}(\alpha_1, \dots, \alpha_{n-2}) \supset \dots \supset A_2(\alpha_1) \supset A_1 \supset A_0.$$

We call such sequences strings. To obtain a standard n-tuple, first pick a string and then select one element $d_{\alpha_{n-1}}$ from the largest set A_{n-1} of the string. (A_{n-1} already depends upon the choice of $d_{\alpha_1}, \dots, d_{\alpha_{n-2}}$.) If an n-tuple $(\alpha) = (d_{\alpha_{n-1}}, \dots, d_{\alpha_1}, 1)$ is chosen in the above way, we write $(\alpha) \subset A$.

In order to establish the converse of theorem (2.3) we require the following lemma.

Lemma (2.4).

If (β) is any n-tuple, $d_{\beta_0} = 1$, then the equivalence class of (β) must contain a standard n-tuple $(\alpha) \subset A$.

Proof.

Suppose the contrary. Choose from the equivalence class of (β) the n-tuple (γ) that agrees most closely with some standard n-tuple $(\alpha) \subset A$ in the sense that $d_{\gamma_i} = d_{\alpha_i}$ $(i = 0, \ldots, r-1)$ but $d_{\gamma_r} \neq d_{\alpha_r}$, where $r < n-1$ and r is maximal. But

$$d_{\gamma_r} \in \boldsymbol{G} = \bigcup \boldsymbol{K}^{\alpha}_{r-1} d_{\alpha_r} \boldsymbol{K}.$$

So $d_{\gamma_r} = p_{r-1} d_{\alpha_r} k$, where $p_{r-1} \in \boldsymbol{K}^{\alpha}_{r-1}$, $k \in \boldsymbol{K}$. Define $d_{\lambda_s} = p^{-1}_{r-1} d_{\gamma_s}$ $(s = r+1, \ldots, n-1)$ then, by (2.1)

$$(\gamma) \sim (d_{\lambda_{n-1}}, \ldots, d_{\lambda_{r+1}}, d_{\alpha_r}, \ldots, d_{\alpha_1}, 1).$$

This contradicts the maximality of r and establishes the lemma.

Theorem (2.5).

Let (β) and (β') be any two n-tuples with $d_{\beta_0} = d_{\beta'_0} = 1$. If $\Omega_{(\beta)} = \Omega_{(\beta')}$ then $(\beta) \sim (\beta')$.

Proof.

By Lemma (2.4), there exist standard n-tuples (α), $(\alpha') \subset A$ such that $(\beta) \sim (\alpha)$ and $(\beta') \sim (\alpha')$. Hence, by theorem (2.3), $\Omega_{(\beta)} = \Omega_{(\alpha)}$ and $\Omega_{(\beta')} = \Omega_{(\alpha')}$. If $(\alpha) \neq (\alpha')$, the direct sum decomposition of Ω shows that $\Omega_{(\alpha)} \neq \Omega_{(\alpha')}$, which contradicts the hypothesis $\Omega_{(\beta)} = \Omega_{(\beta')}$. So $(\beta) \sim (\alpha) = (\alpha') \sim (\beta')$. The result follows since $\sim$ is an equivalence relation.

3. The action of the symmetric group

In § 1, we wrote down a partial decomposition of the carrier space Ω of the nth Kronecker power of $(D \uparrow \boldsymbol{G})$ as a direct sum of $\boldsymbol{G}$-invariant spaces $\Omega_{(\alpha)}$, where

$$\Omega_{(\alpha)} = \sum q^0_{\sigma} q^1_{\sigma'} \ldots q^{n-1}_{\sigma''} (d_{\alpha_{n-1}} \phi_i, \ldots, d_{\alpha_1} \phi_j, \phi_k).$$

In order to obtain all the permutations of the ordered n-tuples of functions spanning $\Omega_{(\alpha)}$, it is sufficient to permute the index set (α), since the set of basis functions of the representation D and the set of left coset representatives of $\boldsymbol{K}_{n-1}$ in $\boldsymbol{G}$ are common to each entry of the n-tuple. It is a corollary of the next theorem that the $n!$ spaces obtained by permuting the entries of (α) are all present in the partial decomposition of Ω.

Theorem (3.1).

The $n!$ spaces

$$\Omega_{(\alpha^{-1}_{\pi(0)} \alpha_{\pi(n-1)}, \ldots, \alpha^{-1}_{\pi(0)} \alpha_{\pi(1)}, 1)} \tag{3.1}$$

where $\pi \in \boldsymbol{S}_n$, $d_{\alpha_0} = 1$, are carrier spaces for equivalent representations of $\boldsymbol{G}$.

Proof.

By equations (1.12) and (1.13), the above space carries the representation Γ_{π} of $\boldsymbol{G}$, where

$$\Gamma_{\pi} = [(D_{\alpha^{-1}_{\pi(0)} \alpha_{\pi(n-1)}} \otimes \ldots \otimes D_{\alpha^{-1}_{\pi(0)} \alpha_{\pi(1)}} \otimes D) \downarrow d^{-1}_{\alpha_{\pi(0)}} \boldsymbol{K}_{n-1} d_{\alpha_{\pi(0)}}] \uparrow \boldsymbol{G}$$

and $K_{n-1} = K_{n-1}(\alpha_1, \ldots, \alpha_{n-1})$. Our notation means that each representation in the Kronecker product is subduced down to $d_{\alpha_{\pi(0)}}^{-1} K_{n-1} d_{\alpha_{\pi(0)}}$. Also $D_{\alpha\beta}$ is a representation of $d_\alpha d_\beta K d_\beta^{-1} d_\alpha^{-1}$ defined by $D_{\alpha\beta}(d_\alpha d_\beta k d_\beta^{-1} d_\alpha^{-1}) = D(k)$, for all $k \in K$. Hence $D_{\alpha\beta} = (D_\beta)_\alpha$.

By the properties of induced representations

$$\begin{aligned}\Gamma_\pi &= \{[(D_{\alpha_{\pi(n-1)}})_{\alpha_{\pi(0)}^{-1}} \otimes \ldots \otimes (D_{\alpha_{\pi(1)}})_{\alpha_{\pi(0)}^{-1}} \otimes D]_{\alpha_{\pi(0)}} \downarrow K_{n-1}\} \uparrow G \\ &= [(D_{\alpha_{\pi(n-1)}} \otimes \ldots \otimes D_{\alpha_{\pi(1)}} \otimes D_{\alpha_{\pi(0)}}) \downarrow K_{n-1}] \uparrow G.\end{aligned}$$

Clearly $(D_{\alpha_{\pi(n-1)}} \otimes \ldots \otimes D_{\alpha_{\pi(0)}})$ and $(D_{\alpha_{n-1}} \otimes \ldots \otimes D_{\alpha_0})$ are equivalent representations of K_{n-1}. Hence their induced representations will be equivalent.

Corollary (3.2).

$$\Omega_{(\alpha_{\pi(0)}^{-1}\alpha_{\pi(n-1)}, \ldots, \alpha_{\pi(0)}^{-1}\alpha_{\pi(1)}, 1)} = \sum q_\sigma^0 q_{\sigma'}^1 \ldots q_{\sigma''}^{n-1}(d_{\alpha_{\pi(n-1)}}\phi_i, \ldots, d_{\alpha_{\pi(0)}}\phi_j).$$

This leads us to define the action of the symmetric group S_n on n-tuples (α) of elements of G. Let

$$\pi : (0, 1, \ldots, n-1) \to (\pi(0), \pi(1), \ldots, \pi(n-1)).$$

Define a map $\hat{\pi} : G \otimes \ldots \otimes G \to G \otimes \ldots \otimes G$ (n times) by $\hat{\pi}(\alpha) = (\beta)$, where

$$d_{\beta_s} = d_{\alpha_{\pi(s)}} \qquad (s = 0, \ldots, n-1). \tag{3.2}$$

It can be shown that $\pi \to \hat{\pi}$ is an anti-homomorphism.

Define the space

$$\Omega_{\hat{\pi}(\alpha)} = \sum q_\sigma^0 \ldots q_{\sigma''}^{n-1}(d_{\alpha_{\pi(n-1)}}\phi_i, \ldots, d_{\alpha_{\pi(0)}}\phi_j). \tag{3.3}$$

This is a carrier space for the representation Γ_π of G, and so it is a suitable generalization of definition (1.13). Note that we must use the form of $\Omega_{\hat{\pi}(\alpha)}$ given by equation (3.1) when using theorem (2.3) or its converse.

Lemma (3.3).

If $(\alpha) \sim (\beta)$, $d_{\alpha_0} = d_{\beta_0} = 1$, then $\Omega_{\hat{\pi}(\alpha)} = \Omega_{\hat{\pi}(\beta)}$ for all $\pi \in S_n$.

Proof.

Using the notation of theorem (2.3)

$$\begin{aligned}\Omega_{(\beta)} &= \sum w_\sigma^0 \ldots w_{\sigma'}^{n-1}(d_{\beta_{n-1}}\phi_i, \ldots, d_{\beta_1}\phi_j, \phi_k) \\ \Omega_{(\alpha)} &= \sum w_\sigma^0 \ldots w_{\sigma'}^{n-1} p_0^{-1} \ldots p_{n-2}^{-1}(d_{\alpha_{n-1}}\phi_i, \ldots, d_{\alpha_1}\phi_j, \phi_k).\end{aligned}$$

Hence

$$\begin{aligned}\Omega_{\hat{\pi}(\beta)} &= \sum w_\sigma^0 \ldots w_{\sigma'}^{n-1}(d_{\beta_{\pi(n-1)}}\phi_i, \ldots, d_{\beta_{\pi(0)}}\phi_k) \\ \Omega_{\hat{\pi}(\alpha)} &= \sum w_\sigma^0 \ldots w_{\sigma'}^{n-1} p_0^{-1} \ldots p_{n-2}^{-1}(d_{\alpha_{\pi(n-1)}}\phi_i, \ldots, d_{\alpha_{\pi(0)}}\phi_k).\end{aligned}$$

The remaining steps of the proof are exactly as in theorem (2.3) and are therefore omitted.

Define the action of S_n on the spaces $\Omega_{(\alpha)}$ as follows:

$$\pi'[\Omega_{(\alpha)}] = \Omega_{\widehat{\pi^{-1}}(\alpha)}. \tag{3.4}$$

It can be shown that $\pi \to \pi'$ is a homomorphism. Define the stability subgroup $\boldsymbol{S}_n(\alpha)$ of (α) by

$$\boldsymbol{S}_n(\alpha) = \{\pi \in \boldsymbol{S}_n : \pi'\Omega_{(\alpha)} = \Omega_{(\alpha)}\}. \tag{3.5}$$

Clearly this is a subgroup of $\boldsymbol{S}_n$.

Lemma (3.4).

(i) If $\tau'\Omega_{(\alpha)} = \Omega_{(\beta)}$, then $\boldsymbol{S}_n(\beta) = \tau\boldsymbol{S}_n(\alpha)\tau^{-1}$.

(ii) Among the $n!$ spaces $\pi'\Omega_{(\alpha)}$, $\pi \in \boldsymbol{S}_n$, there are exactly $n!/t$ distinct spaces, where $t = |\boldsymbol{S}_n(\alpha)|$.

This follows from the general result about the transitive action of a group on a space. The spaces of lemma (3.4) (ii) are said to form an orbit under $\boldsymbol{S}_n$ which can be seen to be invariant under both $\boldsymbol{G}$ and $\boldsymbol{S}_n$. Now define

$$\boldsymbol{E}_n(\alpha) = \{\pi \in \boldsymbol{S}_n : \hat{\pi}(\alpha) = (\alpha)\}. \tag{3.6}$$

This is a subgroup of $\boldsymbol{S}_n(\alpha)$ and it is nontrivial only if some of the double coset representatives are identical. It follows from the next theorem, that $\boldsymbol{E}_n(\alpha)$ is a normal subgroup of $\boldsymbol{S}_n(\alpha)$ so long as (α) is a standard n-tuple.

Theorem (3.5).

Let $(\alpha) \subset A$, $\pi \in \boldsymbol{S}_n(\alpha)$. The entries in the i, j positions of (α) are equal if and only if the entries in the $\pi(i)$, $\pi(j)$ positions are also equal $(i, j = 0, \ldots, n-1)$.

Proof.

$\Omega_{(\alpha)} = \Omega_{\hat{\pi}(\alpha)}$ so by theorem (2.5) and equation (3.1)

$$d_{\alpha_r} = p_{r-1} \ldots p_0 d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(r)}} k_{r-1} \qquad (r = 1, \ldots, n-1)$$

where $p_i \in \boldsymbol{K}^{\alpha}_i$, $k_i \in \boldsymbol{K}$ $(i = 0, \ldots, n-2)$. Suppose $d_{\alpha_i} = d_{\alpha_j}$ $(i < j)$, then

$$d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(i)}} = (p_{i-1} \ldots p_0)^{-1}(p_{j-1} \ldots p_i)(p_{i-1} \ldots p_0) d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(j)}} k_{j-1} k^{-1}_{i-1}. \tag{3.7}$$

By lemma (2.1)

$$(p_{i-1} \ldots p_0)^{-1}(p_{j-1} \ldots p_i)(p_{i-1} \ldots p_0) \in d^{-1}_{\alpha_{\pi(0)}} \boldsymbol{K}^{\hat{\pi}(\alpha)}_i d_{\alpha_{\pi(0)}} \subset d^{-1}_{\alpha_{\pi(0)}} d_{\alpha_{\pi(i)}} \boldsymbol{K} d^{-1}_{\alpha_{\pi(i)}} d_{\alpha_{\pi(0)}}.$$

Substituting in equation (3.7) we obtain $d_{\alpha_{\pi(i)}} \in d_{\alpha_{\pi(j)}}\boldsymbol{K}$. But (α) is a standard n-tuple so $d_{\alpha_{\pi(i)}} = d_{\alpha_{\pi(j)}}$. Clearly the converse also holds.

From this point we shall assume that (α) is a standard n-tuple. Theorem (3.5) gives $\boldsymbol{E}_n(\alpha) = \boldsymbol{E}_n(\hat{\pi}(\alpha))$ for all $\pi \in \boldsymbol{S}_n(\alpha)$ and hence $\boldsymbol{E}_n(\alpha)$ is a normal subgroup of $\boldsymbol{S}_n(\alpha)$. Define the factor group

$$\boldsymbol{F}_n(\alpha) = \boldsymbol{S}_n(\alpha)/\boldsymbol{E}_n(\alpha). \tag{3.8}$$

When $\boldsymbol{S}_n(\alpha)$ is nontrivial, not all the permutations of the basis for the representation $(D_{\alpha_{n-1}} \otimes \ldots \otimes D_{\alpha_1} \otimes D) \downarrow \boldsymbol{K}_{n-1}$ are explicitly present in the basis for the space $T(\alpha)$. However we show that with each element $\pi \in \boldsymbol{S}_n(\alpha)$ we can associate an element $a_\pi \in \boldsymbol{G}$. The group $\boldsymbol{M}$, generated by $\boldsymbol{K}_{n-1}$ and the set $\{a_\pi : \pi \in \boldsymbol{S}_n(\alpha)\}$, will play an important role in the determination of the required decomposition.

Theorem (3.6).

Let $(\alpha) \subset A$, then $\pi \in \boldsymbol{S}_n(\alpha)$ if and only if there exists

$$a_\pi \in \bigcap_{s=0}^{n-1} d_{\alpha_{\pi(s)}} \boldsymbol{K} d_{\alpha_s}^{-1}.$$

Proof.

Let $\sigma = \pi^{-1}$, then $\pi \in \boldsymbol{S}_n(\alpha)$ implies $\Omega_{(\alpha)} = \Omega_{\hat{\sigma}(\alpha)}$. By theorem (2.5)

$$d_{\alpha_r} = p_{r-1} \dots p_0 d_{\alpha_{\sigma(0)}}^{-1} d_{\alpha_{\sigma(r)}} k_{r-1}, \qquad (r = 1, \dots, n-1), \tag{3.9}$$

where $p_i \in \boldsymbol{K}_i$, $k_i \in \boldsymbol{K}$ $(i = 0, \dots, n-2)$. Putting $r = n-1$

$$\boldsymbol{K}_{n-2} d_{\alpha_{n-1}} \boldsymbol{K} = \boldsymbol{K}_{n-2} p_{n-3} \dots p_0 d_{\alpha_{\sigma(0)}}^{-1} d_{\alpha_{\sigma(n-1)}} \boldsymbol{K}.$$

That is

$$\begin{aligned}
&\boldsymbol{K}_{n-2} p_{n-3} \dots p_0 (d_{\alpha_{\sigma(0)}}^{-1} d_{\alpha_{\sigma(n-1)}} \boldsymbol{K} d_{\alpha_{\sigma(n-1)}}^{-1} d_{\alpha_{\sigma(0)}}) \\
&\quad = \boldsymbol{K}_{n-2} d_{\alpha_{n-1}} d_{\alpha_{\sigma(n-1)}}^{-1} d_{\alpha_{\sigma(0)}} (d_{\alpha_{\sigma(0)}}^{-1} d_{\alpha_{\sigma(n-1)}} \boldsymbol{K} d_{\alpha_{\sigma(n-1)}}^{-1} d_{\alpha_{\sigma(0)}}).
\end{aligned}$$

From the properties of double cosets, it follows that

$$\boldsymbol{K}_{n-2} p_{n-3} \dots p_0 \cap d_{\alpha_{n-1}} d_{\alpha_{\sigma(n-1)}}^{-1} d_{\alpha_{\sigma(0)}} (d_{\alpha_{\sigma(0)}}^{-1} d_{\alpha_{\sigma(n-1)}} \boldsymbol{K} d_{\alpha_{\sigma(n-1)}}^{-1} d_{\alpha_{\sigma(0)}})$$

is non-empty and hence contains an element a'. Let

$$a_\pi = a' d_{\alpha_{\sigma(0)}}^{-1} \in \boldsymbol{K}_{n-2} p_{n-3} \dots p_0 d_{\alpha_{\sigma(0)}}^{-1} \cap d_{\alpha_{n-1}} \boldsymbol{K} d_{\alpha_{\sigma(n-1)}}^{-1}.$$

If $i \geqslant r$, then $p_i \in \boldsymbol{K}_r \subset d_{\alpha_r} \boldsymbol{K} d_{\alpha_r}^{-1}$, so

$$\begin{aligned}
d_{\alpha_r} \boldsymbol{K} d_{\alpha_r}^{-1} p_{n-3} \dots p_0 d_{\alpha_{\sigma(0)}}^{-1} \qquad & (r = 0, \dots, n-2) \\
= d_{\alpha_r} \boldsymbol{K} d_{\alpha_r}^{-1} p_{r-1} \dots p_0 d_{\alpha_{\sigma(0)}}^{-1} & \\
= d_{\alpha_r} \boldsymbol{K} d_{\alpha_{\sigma(r)}}^{-1}. &
\end{aligned}$$

The last line follows from equation (3.9), hence

$$a_\pi \in \bigcap_{s=0}^{n-1} d_{\alpha_{\pi(s)}} \boldsymbol{K} d_{\alpha_s}^{-1}.$$

Conversely, $a_\pi = d_{\alpha_{\pi(s)}} k_s d_{\alpha_s}^{-1}$ $(s = 0, \dots, n-1)$, where $k_s \in \boldsymbol{K}$, so

$$d_{\alpha_{\pi(0)}}^{-1} d_{\alpha_{\pi(s)}} = d_{\alpha_{\pi(0)}}^{-1} a_\pi d_{\alpha_s} k_s^{-1} = k_0 d_{\alpha_s} k_s^{-1}.$$

Hence, by theorem (2.3) $\Omega_{(\alpha)} = \Omega_{\hat{\pi}(\alpha)}$, and so $\pi \in \boldsymbol{S}_n(\alpha)$.

Theorem (3.7).

Let $(\alpha) \subset A$. With each $\pi \in \boldsymbol{S}_n(\alpha)$ associate an element $a_\pi \in \bigcap_{s=0}^{n-1} d_{\alpha_{\pi(s)}} \boldsymbol{K} d_{\alpha_s}^{-1}$. Let $\boldsymbol{M}$ be the group generated by $\boldsymbol{K}_{n-1}$ and the set $\{a_\pi : \pi \in \boldsymbol{S}_n(\alpha)\}$, then there is an epimorphism $\phi_\alpha : \boldsymbol{S}_n(\alpha) \to \boldsymbol{M}/\boldsymbol{K}_{n-1}$ given by $\phi_\alpha(\pi) = a_\pi \boldsymbol{K}_{n-1}$, with kernel $\boldsymbol{E}_n(\alpha)$.

Proof.

First note that if $a_\pi, a'_\pi \in \bigcap_{s=0}^{n-1} d_{\alpha_{\pi(s)}} K d_{\alpha_s}^{-1}$ then $a'_\pi \in a_\pi K_{n-1}$. Now, for each $\pi \in S_n(\alpha)$,

$$a_\pi K_{n-1} a_\pi^{-1} = a_\pi \left(\bigcap_{r=0}^{n-1} d_{\alpha_r} K d_{\alpha_r}^{-1} \right) a_\pi^{-1}.$$

But $a_\pi d_{\alpha_s} \in d_{\alpha_{\pi(s)}} K$, $(s = 0, \ldots, n-1)$, and hence K_{n-1} is a normal subgroup of M.

Let $\phi_\alpha(\pi) = a_\pi K_{n-1}$, $\phi_\alpha(\sigma) = a_\sigma K_{n-1}$ where π, $\sigma \in S_n(\alpha)$. Then

$$a_\pi a_\sigma \in \left(\bigcap_{s=0}^{n-1} d_{\alpha_{\pi(s)}} K d_{\alpha_s}^{-1} \right) \left(\bigcap_{t=0}^{n-1} d_{\alpha_{\sigma(t)}} K d_{\alpha_t}^{-1} \right) = \bigcap_{s=0}^{n-1} d_{\alpha_{\pi\sigma(s)}} K d_{\alpha_s}^{-1}.$$

So

$$\begin{aligned} \phi_\alpha(\pi\sigma) &= a_\pi a_\sigma K_{n-1} \\ &= a_\pi K_{n-1} a_\sigma K_{n-1} \\ &= \phi_\alpha(\pi)\phi_\alpha(\sigma). \end{aligned}$$

By the converse of theorem (3.6), ϕ_α is subjective, and hence an epimorphism.

Suppose $\pi \in \operatorname{Ker} \phi_\alpha$, then $a_\pi \in K_{n-1}$. So $d_{\alpha_r} k_r d_{\alpha_r}^{-1} = d_{\alpha_{\pi(r)}} k'_r d_{\alpha_r}^{-1}$ $(r = 0, \ldots, n-1)$, where $k_r, k'_r \in K$. Hence $d_{\alpha_r} \in d_{\alpha_{\pi(r)}} K$. But (α) is a standard n-tuple so $\hat{\pi}(\alpha) = (\alpha)$ and $\pi \in E_n(\alpha)$. Clearly $\pi \in E_n(\alpha)$ implies $\pi \in \operatorname{Ker} \phi_\alpha$ and so $F_n(\alpha)$ is isomorphic to M/K_{n-1}.

4. Properties of standard *n*-tuples

We now prove a few technical lemmas about the properties of the standard n-tuples introduced earlier, and then interpret their meaning.

Lemma (4.1).

Let $(\alpha) \sim (\beta)$, where $(\alpha) \subset A$ and (β) is an arbitrary n-tuple with $d_{\beta_0} = 1$. Then, the entries in the r, s positions of (β) are equal implies that the entries in the r, s positions of (α) are equal $(r, s = 0, \ldots, n-1)$.

Proof.

Suppose $d_{\beta_s} = d_{\beta_r}$, $s < r$. Since $(\alpha) \sim (\beta)$

$$d_{\alpha_t} = p_{t-1} \cdots p_0 d_{\beta_t} k_{t-1}, \qquad (t = 1, \ldots, n-1),$$

where $p_i \in K_i^\alpha$, $k_i \in K$ $(i = 1, \ldots, n-2)$. Also $d_{\alpha_0} = d_{\beta_0} = 1$. In particular

$$\begin{aligned} d_{\alpha_r} &= p_{r-1} \cdots p_0 d_{\beta_r} k_{r-1} \\ &= p_{r-1} \cdots p_s d_{\alpha_s} k_{s-1}^{-1} k_{r-1}. \end{aligned}$$

But $p_{r-1} \cdots p_s \in K_s^\alpha \subset d_{\alpha_s} K d_{\alpha_s}^{-1}$, $(s = 0, \ldots, r-1)$. Hence $d_{\alpha_r} \in d_{\alpha_s} K$, and since (α) is a standard n-tuple $d_{\alpha_r} = d_{\alpha_s}$.

Corollary (4.2).

Let B be an alternative set of standard n-tuples arising from the non-uniqueness of the prescription for obtaining A, and suppose $(\alpha) \sim (\beta)$ where $(\alpha) \subset A$, $(\beta) \subset B$. Then, the entries in the r, s positions of (α) are equal if and only if the entries in the r, s positions of (β) are equal $(r, s = 0, \ldots, n-1)$.

We say the n-tuple (α) is of type $(\lambda_1, \ldots, \lambda_r)$ if (α) has λ_1 entries equal to $d_\alpha^{(1)}, \ldots, \lambda_r$ entries equal to $d_\alpha^{(r)}$, where $\Sigma_{i=1}^r \lambda_i = n$.

Lemma (4.3).

Let $(\alpha) \subset A$ be of type $(\lambda_1, \ldots, \lambda_r)$. If $\Omega_{\hat{\pi}(\alpha)} = \Omega_{(\beta)}$ where $(\beta) \subset A$, then (β) is also of type $(\lambda_1, \ldots, \lambda_r)$.

Proof.

By lemma (4.1) and since $(\beta) \subset A$:

Number of d_β equal to d_{β_s} in (β)

$$\geqslant \text{number of entries equal to } d_{\alpha_{\pi(0)}}^{-1} d_{\alpha_{\pi(s)}}$$

$$= \text{number of entries equal to } d_{\alpha_{\pi(s)}} \text{ in } (\alpha).$$

$\Omega_{\hat{\pi}(\alpha)} = \Omega_{(\beta)}$ implies $\Omega_{(\alpha)} = \Omega_{\widehat{\pi^{-1}}(\beta)}$ and so we may reverse the argument:

Number of d_α equal to d_{α_t} in (α)

$$\geqslant \text{number of entries equal to } d_{\beta_{\pi^{-1}(0)}}^{-1} d_{\beta_{\pi^{-1}(t)}}$$

$$= \text{number of entries equal to } d_{\beta_{\pi^{-1}(t)}} \text{ in } (\beta).$$

The result follows by choosing $t = \pi(s)$.

Now we must interpret these results. If (α) is of type $(\lambda_1, \ldots, \lambda_r)$ then $\Omega_{(\alpha)}$ may be regarded as the tensor product of r tensor subspaces, each indexed by a distinct double coset representative $d_\alpha^{(r)}$. Lemma (4.1) shows that r is minimal if $(\alpha) \subset A$ is a standard n-tuple and corollary (4.2) shows that this decomposition is independent of the particular choice of A. Lemma (4.3) shows that this decomposition is not affected if, within each orbit, we consider the n-tuples $\hat{\pi}(\alpha)$, $\pi \in \boldsymbol{S}_n$, rather than the standard n-tuples. In order to decompose the nth Kronecker power of $(D \uparrow \boldsymbol{G})$ into its symmetrized parts, the results of §§ 5 and 6 should be applied to one standard n-tuple from each orbit under $\boldsymbol{S}_n$.

5. Distinct double coset representatives

As a preliminary to the general case, we consider a standard n-tuple $(\alpha) = (d_{\alpha_{n-1}}, \ldots, d_{\alpha_1}, 1)$ with no two entries the same, that is $d_{\alpha_i} \neq d_{\alpha_j} (0 \leqslant i < j \leqslant n-1)$. In this case $\boldsymbol{E}_n(\alpha) = \{1\}$ and so if $\pi \in \boldsymbol{S}_n(\alpha)$, by theorem (3.7), there is an isomorphism $\phi_\alpha : \pi \to a_\pi \boldsymbol{K}_{n-1}$.

Let $\Delta = D_{\alpha_{n-1}} \otimes D_{\alpha_{n-2}} \otimes \ldots \otimes D$ be the UR of $\boldsymbol{K}_{n-1}$ with basis set

$$\{(d_{\alpha_{n-1}}\phi_i, \ldots, d_{\alpha_1}\phi_j, \phi_k) : i, j, k = 1, \ldots, d\}.$$

Define P to be the $d^n \times d^n$ unitary matrix with components

$$P_{i_{n-1}i_{n-2}\ldots i_0, j_{n-1}j_{n-2}\ldots j_0} = \prod_{s=0}^{n-1} D(d_{\alpha_{\pi(s)}}^{-1} a_\pi d_{\alpha_s})_{i_s j_s} \tag{5.1}$$

where $\pi \in S_n(\alpha)$ and $d_{\alpha_0} = 1$. For convenience we shorten the left-hand side of equation (5.1) to P_{ij}. Then

$$a_\pi(d_{\alpha_{n-1}}\phi_{i_{n-1}}, \ldots, d_{\alpha_0}\phi_{i_0})P_{ij}^{-1} = (d_{\alpha_{\pi(n-1)}}\phi_{j_{n-1}}, \ldots, d_{\alpha_{\pi(0)}}\phi_{j_0}) \tag{5.2}$$

where the summation convention is being used.

Let $a_\pi = b_\pi l \in \boldsymbol{M}$, where $l \in \boldsymbol{K}_{n-1}$, and define

$$Q_{ij} = \prod_{s=0}^{n-1} D(d_{\alpha_{\pi(s)}}^{-1} b_\pi d_{\alpha_s})_{i_s j_s}.$$

Then $P_{ij} = Q_{ik}\Delta(l)_{kj}$. Hence if we write

$$\langle d_\alpha \phi|_i = (d_{\alpha_{n-1}}\phi_{i_{n-1}}, \ldots, d_{\alpha_1}\phi_{i_1}, d_{\alpha_0}\phi_{i_0}) \tag{5.3}$$

then

$$\begin{aligned}(a_\pi \langle d_\alpha \phi | P^{-1})_i &= b_\pi l \langle d_\alpha \phi|_j P_{ji}^{-1} \\ &= b_\pi \langle d_\alpha \phi|_k \Delta(l)_{kj} P_{ji}^{-1} \\ &= (b_\pi \langle d_\alpha \phi | Q^{-1})_i.\end{aligned}$$

So we have the following result.

Lemma (5.1).

The action of $a_\pi \ldots P^{-1}$ on $\langle d_\alpha \phi|_i$ is independent of the chosen coset representative of $\boldsymbol{K}_{n-1}$ in $\boldsymbol{M}$.

Now define the operator $a_\pi \ldots P(\pi)^{-1}$ by

$$a_\pi \langle d_\alpha \phi|_j P(\pi)^{-1} = a_\pi \langle d_\alpha \phi|_i P_{i\pi(j)}^{-1} \tag{5.4}$$

where $\pi(j)$ means $j_{\pi(n-1)} j_{\pi(n-2)} \cdots j_{\pi(0)}$, then from (5.2) the right-hand side of equation (5.4) becomes $(d_{\alpha_{\pi(n-1)}}\phi_{j_{\pi(n-1)}}, \ldots, d_{\alpha_{\pi(0)}}\phi_{j_{\pi(0)}})$. We denote this basis vector by $\langle d_\alpha \phi|_{\pi(j)}$. Hence we see that

$$a_\pi \langle d_\alpha \phi|_j P(\pi)^{-1} = \langle d_\alpha \phi|_{\pi(j)}. \tag{5.5}$$

Also it may be proved that, for all $\sigma \in \boldsymbol{S}_n$,

$$\begin{aligned}a_\pi \langle d_\alpha \phi|_{\sigma(i)} P(\pi)^{-1} &= a_\pi \langle d_\alpha \phi|_{\sigma(j)} P(\pi)_{ji}^{-1} \\ &= \langle d_\alpha \phi|_{\pi\sigma(i)}.\end{aligned} \tag{5.6}$$

These are important results and may be summarized by saying that the operator $a_\pi \ldots P(\pi)^{-1}$ permutes the components of the basis vectors of the carrier space of Δ.

Equation (5.6) leads to the definition of another action of the symmetric group, which we write as $\bar{\sigma} \in \bar{\boldsymbol{S}}_n$:

$$\bar{\sigma}\langle d_\alpha \phi|_{\tau(j)} = \langle d_\alpha \phi|_{\sigma\tau(j)} \tag{5.7}$$

for all $\tau \in \boldsymbol{S}_n$. The bars distinguish this action from the natural action of $\boldsymbol{S}_n$ on a tensor space; namely, if $\sigma \in \boldsymbol{S}_n$, then

$$\sigma\langle d_\alpha \phi|_{\tau(j)} = \langle d_\alpha \phi|_{\tau\sigma^{-1}(j)}. \tag{5.8}$$

Comparing equations (5.6) and (5.7) we see that we may identify the operator $a_\pi \ldots P(\pi)^{-1}$ with $\bar{\pi} \in \bar{\boldsymbol{S}}_n$.

Lemma (5.2).

Let $\pi \in \boldsymbol{S}_n(\alpha)$, then the map $\psi_\alpha : \pi \to a_\pi \ldots P(\pi)^{-1}$ is a monomorphism of $\boldsymbol{S}_n(\alpha)$ into $\bar{\boldsymbol{S}}_n$.

Proof.

Let $\pi, \tau \in \boldsymbol{S}_n(\alpha)$, then

$$\begin{aligned}\psi_\alpha(\pi)\psi_\alpha(\tau)\langle d_\alpha\phi|_j &= \psi_\alpha(\pi)\langle d_\alpha\phi|_{\tau(j)} \\ &= \langle d_\alpha\phi|_{\pi\tau(j)} \\ &= \psi_\alpha(\pi\tau)\langle d_\alpha\phi|_j.\end{aligned}$$

Also $\psi_\alpha(\pi)\langle d_\alpha\phi|_j = \langle d_\alpha\phi|_j$, for all j, if and only if $\pi = 1$. Hence ψ_α is a monomorphism.

The following lemma will be required later and is proved using equations (5.6) and (5.8).

Lemma (5.3).

Let $a_\pi \in \boldsymbol{M}$ and $\sigma \in \boldsymbol{S}_n$, then

$$a_\pi\sigma\langle d_\alpha\phi|_{\tau(i)} = \sigma a_\pi\langle d_\alpha\phi|_{\tau(i)},$$

for all $\tau \in \boldsymbol{S}_n$.

Define $V_{\hat{\tau}(\alpha)}$ to be the carrier space of the UR $(D_{\alpha_{\tau(n-1)}} \otimes \ldots \otimes D_{\alpha_{\tau(0)}}) \downarrow \boldsymbol{K}_{n-1}$ with basis $\{\langle d_\alpha\phi|_{\tau(i)} : \text{for all } i\}$. Consider the coset decomposition

$$\boldsymbol{S}_n = \bigcup_i \tau_i \boldsymbol{S}_n(\alpha) = \bigcup_i \boldsymbol{S}_n(\alpha)\tau_i^{-1}. \tag{5.9}$$

By lemma (3.4), the spaces $\tau_i'\Omega_{(\alpha)}$ are distinct and so we induce the representation of $\boldsymbol{K}_{n-1}$ carried by $\oplus_i V_{\widehat{\tau_i^{-1}}(\alpha)}$ up to $\boldsymbol{M}$. The carrier space of this induced representation, which we denote by $W(\alpha)$, has basis $\{a_\pi\tau_i\langle d_\alpha\phi|_j : \text{for all } j, \tau_i \text{ and } \pi \in \boldsymbol{S}_n(\alpha)\}$. Change to the unitarily equivalent basis $\{a_\pi\tau_i\langle d_\alpha\phi|_j P(\pi)^{-1}\} = \{\tau_i\pi^{-1}\langle d_\alpha\phi|_j\}$. For fixed $\pi \in \boldsymbol{S}_n(\alpha)$ and $\tau_i \in \boldsymbol{S}_n$, this is a basis for the space $V_{\widehat{\pi\tau_i^{-1}}(\alpha)}$ and so, by virtue of (5.9), $W(\alpha)$ has basis $\{\sigma\langle d_\alpha\phi|_i : \text{for all } i, \text{for all } \sigma \in \boldsymbol{S}_n\}$. Hence, $W(\alpha)$ is invariant under both $\boldsymbol{M}$ and $\boldsymbol{S}_n$, and contains explicitly all the permutations of the basis vectors of the carrier space of Δ.

The action of $\boldsymbol{M}$ on $W(\alpha)$ is given by:

$$\begin{aligned}a_\pi\langle d_\alpha\phi|_{\sigma(i)} &= \langle d_\alpha\phi|_{\pi\sigma(j)}P(\pi)_{ji} \\ &= \bar{\pi}\langle d_\alpha\phi|_{\sigma(j)}P(\pi)_{ji}.\end{aligned} \tag{5.10}$$

If $\pi = 1$, then $a_\pi = l \in \boldsymbol{K}_{n-1}$ and $P(1) = \Delta(l)$. This action is independent of the choice of $\sigma \in \boldsymbol{S}_n$ and so, for fixed i, we may replace the functions $\langle d_\alpha\phi|_{\sigma(i)}$, for all $\sigma \in \boldsymbol{S}_n$, by an equivalent set $\psi_{si}(s = 1, \ldots, n!)$ which is a basis for the regular representation of $\bar{\boldsymbol{S}}_n$ in fully reduced form. To preserve continuity, a proof that $\boldsymbol{S}_n$ and $\bar{\boldsymbol{S}}_n$ give rise to the same symmetry classes will be deferred to the end of this section.

Let $f_\nu = \dim\,[\nu]$, then

$$\bar{\pi}\psi_{si} = \psi_{ti}\left(\bigoplus_\nu f_\nu[\nu](\bar{\pi})\right)_{ts}. \tag{5.11}$$

Hence, if $l \in \boldsymbol{K}_{n-1}$, $a_\pi \in \boldsymbol{M}$

$$\begin{aligned} l\psi_{si} &= \psi_{sj}\Delta(l)_{ji} \\ a_\pi \psi_{si} &= \bar{\pi}\psi_{sj}P(\pi)_{ji} \\ &= \psi_{jt}P(\pi)_{ji}\left(\bigoplus_\nu f_\nu[\nu](\bar{\pi})\right)_{ts}. \end{aligned} \tag{5.12}$$

The space $W(\alpha)$ carries the representation Γ of $\boldsymbol{M}$, where

$$\begin{aligned} \Gamma(l) &= 1_{n!} \otimes \Delta(l) \\ \Gamma(a_\pi) &= \bigoplus_\nu (f_\nu[\nu](\bar{\pi}) \otimes P(\pi)). \end{aligned} \tag{5.13}$$

But $n! = \Sigma_\nu f_\nu^2$, so the symmetrized basis of $W(\alpha)$ corresponding to the UR $[\nu]$ of $\bar{\boldsymbol{S}}_n$ carries the representation

$$\begin{aligned} \Gamma_{[\nu]}(l) &= 1_{f_\nu^2} \otimes \Delta(l) \\ \Gamma_{[\nu]}(a_\pi) &= f_\nu[\nu](\bar{\pi}) \otimes P(\pi). \end{aligned} \tag{5.14}$$

In particular

$$\begin{aligned} \Gamma_{[n]}(l) &= \Gamma_{[1^n]}(l) = \Delta(l) \\ \Gamma_{[n]}(a_\pi) &= (-1)^\pi \Gamma_{[1^n]}(a_\pi) = P(\pi) \end{aligned} \tag{5.15}$$

are the representations of $\boldsymbol{M}$ corresponding to the identity representation $[n]$ and the alternating representation $[1^n]$ of $\bar{\boldsymbol{S}}_n$, where $(-1)^\pi$ is the parity of $\bar{\pi} \in \bar{\boldsymbol{S}}_n$.

The carrier space of $\Gamma \uparrow \boldsymbol{G}$ is

$$T(\alpha) = \bigoplus_i \Omega_{\widehat{\tau_i^{-1}(\alpha)}} \tag{5.16}$$

where $\tau_i \in \boldsymbol{S}_n$ is defined by equation (5.9). The space $T(\alpha)$ is invariant under both $\boldsymbol{G}$ and $\boldsymbol{S}_n$ and so may be decomposed

$$T(\alpha) = \bigoplus_\nu T(\alpha)^\nu \tag{5.17}$$

where $T(\alpha)^\nu$ is a subspace of $\boldsymbol{\Omega}^\nu$. The UR $\Gamma_{[\nu]} \uparrow \boldsymbol{G}$ is defined on the space $T(\alpha)^\nu$.

The characters in $\boldsymbol{M}$ may be found in the following way. Express π as a product of disjoint cycles, including 1-cycles, in the form

$$\pi = (j_1 \ldots j_{p_1})(j_{p_1+1} \ldots j_{p_2}) \ldots (j_{p_{m-1}+1} \ldots j_n).$$

Then the character of $\Gamma_{[\nu]}(a_\pi)$ is

$$\chi_\nu(a_\pi) = f_\nu \chi_{[\nu]}(\pi) \sum_i P_{\pi(i)i}$$

where $\chi_{[\nu]}$ is the character of the UR $[\nu]$.

$$\begin{aligned} \chi_\nu(a_\pi) &= f_\nu \chi_{[\nu]}(\pi) \sum_i \prod_{s=0}^{n-1} D(d_{\alpha_{\pi(s)}}^{-1} a_\pi d_{\alpha_s})_{i_{\pi(s)} i_s} \\ &= f_\nu \chi_{[\nu]}(\pi) \prod_{r=1}^{m} \chi_{\gamma_r}(a_\pi^{p_r - p_{r-1}}) \end{aligned} \tag{5.18}$$

where $\gamma = \alpha_{j_p}$, $p_m = n$ and $p_0 = 0$. Also

$$\chi_D(k) = \chi_{\gamma_r}(d_{\gamma_r} k d_{\gamma_r}^{-1}), \qquad k \in \boldsymbol{K}$$

where χ_D is the character of the UR D of $\boldsymbol{K}$. In particular

$$\chi_\nu(l) = f_\nu^2 \chi_\Delta(l), \qquad l \in \boldsymbol{K}_{n-1} \tag{5.19}$$

where χ_Δ is the character of the UR Δ.

We shall now show that the actions of $\boldsymbol{S}_n$ and $\bar{\boldsymbol{S}}_n$ on $W(\alpha)$ give rise to the same symmetry classes of $W(\alpha)$. Here, by symmetry class, we mean the set of all functions which belong to a given representation, say $[\nu]$, of $\boldsymbol{S}_n$. $W(\alpha)$ has basis $\{\langle d_\alpha \phi|_{\sigma(i)}$: for all i and $\sigma \in \boldsymbol{S}_n\}$. The projection operator onto Ω^ν is given by

$$\mathscr{P}^\nu = \frac{f_\nu}{n!} \sum_{\tau \in \boldsymbol{S}_n} \chi_{[\nu]}(\tau)\tau, \tag{5.20}$$

since all the characters are real. Hence the space $W(\alpha)^\nu$ is spanned by the functions

$$\begin{aligned}
\mathscr{P}^\nu \langle d_\alpha \phi|_{\sigma(i)} &= \frac{f_\nu}{n!} \sum_{\tau \in \boldsymbol{S}_n} \chi_{[\nu]}(\tau) \langle d_\alpha \phi|_{\sigma\tau^{-1}(i)} \\
&= \frac{f_\nu}{n!} \sum_{\tau \in \boldsymbol{S}_n} \chi_{[\nu]}(\sigma\tau\sigma^{-1}) \overline{\sigma\tau^{-1}\sigma^{-1}} \langle d_\alpha \phi|_{\sigma(i)} \\
&= \frac{f_\nu}{n!} \sum_{\tau \in \boldsymbol{S}_n} \chi_{[\nu]}(\tau) \overline{\tau^{-1}} \langle d_\alpha \phi|_{\sigma(i)} \\
&= \frac{f_\nu}{n!} \sum_{\bar{\tau} \in \bar{\boldsymbol{S}}_n} \chi_{[\nu]}(\bar{\tau}) \bar{\tau} \langle d_\alpha \phi|_{\sigma(i)}.
\end{aligned}$$

The second line follows because $\chi_{[\nu]}$ is a class function and the fourth because π, π^{-1} are conjugate and $\boldsymbol{S}_n, \bar{\boldsymbol{S}}_n$ have the same characters, being different actions of the same group. Now because Ω is a tensor space, the actions of $\boldsymbol{G}$ and $\boldsymbol{S}_n$ commute, and so operating on a symmetrized basis element of $W(\alpha)$ with an element of $\boldsymbol{G}$, as in the construction of a basis for the induced representation, does not change the symmetry of the element.

6. Decomposition into symmetry classes

Let (α) be a standard n-tuple of type $(\lambda_1, \ldots, \lambda_r)$. Without loss of generality, we may group together equal elements and take $(\alpha) = (d_\alpha^{(1)}, \ldots, d_\alpha^{(1)}, d_\alpha^{(2)}, \ldots, d_\alpha^{(r)})$, where $d_\alpha^{(r)} = 1$, since this lies in the orbit under $\boldsymbol{S}_n$. As in § 5, $V_{(\alpha)}$ has basis $\{(d_\alpha^{(1)}\phi_{i_{n-1}}, \ldots, d_\alpha^{(r)}\phi_{i_0})$: for all $i\}$ which is invariant under $\boldsymbol{S}_{\lambda_1} \times \boldsymbol{S}_{\lambda_2} \times \ldots \times \boldsymbol{S}_{\lambda_r}$. Hence each tensor space, indexed by a fixed double coset representative $d_\alpha^{(t)}$, $(t = 1, \ldots, r)$, will split under the action of $\boldsymbol{S}_{\lambda_t}$ and $\boldsymbol{K}_{n-1}$, into symmetry classes, which we denote by R_{μ_t}, and which arise in a manner analogous to that described in the introduction.

Write $d_\alpha^{(t)} = d_{\alpha_t}$ $(t = 1, \ldots, r)$. Let $\tilde{\Delta} = \langle \mu_1 \rangle_{\alpha_1} \otimes \ldots \otimes \langle \mu_r \rangle_{\alpha_r}$ be a UR of $\boldsymbol{K}_{n-1}$, where $\langle \mu_t \rangle_{\alpha_t}$ is some UR of $d_{\alpha_t} \boldsymbol{K} d_{\alpha_t}^{-1}$ on the tensor space indexed by d_{α_t}, corresponding to the UIR $[\mu_t]$ of $\boldsymbol{S}_{\lambda_t}$,

$$\langle \mu_t \rangle_{\alpha_t}(x) = \langle \mu_t \rangle (d_{\alpha_t}^{-1} x d_{\alpha_t}), \tag{6.1}$$

for all $x \in d_{\alpha_t} \boldsymbol{K} d_{\alpha_t}^{-1}$. The UR $\tilde{\Delta}$ is a subrepresentation of the UR Δ defined in § 5. Let

$\{\psi^t_{iw}: i = 1, \ldots, d_{\mu_t}\}$, for each w ($w = 1, \ldots, f_{\mu_t}$), be a basis for the UR $\langle\mu_t\rangle$ of $\boldsymbol{K}$ ($t = 1, \ldots, r$). If $k \in \boldsymbol{K}$ and $\pi \in \boldsymbol{S}_{\lambda_t}$ then

$$\left.\begin{aligned} k\psi^t_{iw} &= \psi^t_{jw}\langle\mu_t\rangle(k)_{ji} \\ \pi\psi^t_{iw} &= \psi^t_{iq}[\mu_t](\pi)_{qw} \end{aligned}\right\}. \tag{6.2}$$

Each UR $\langle\mu_t\rangle$ is a subrepresentation of the λ_tth tensor power of D and so ψ^t_{iw} is a linear combination of λ_t-tuples. Also, the set $\{\psi^t_{iw}: i = 1, \ldots, d_{\mu_t}, w = 1, \ldots, f_{\mu_t}\}$ is a basis for R_{μ_t} and so the upper index, t, denotes the symmetry class to which the function belongs.

By theorem (3.7), there is an isomorphism $\bar{\Phi}_\alpha: \boldsymbol{F}_n(\alpha) \to \boldsymbol{M}/\boldsymbol{K}_{n-1}$ defined by

$$\bar{\Phi}_\alpha: \pi\boldsymbol{E}_n(\alpha) \to a_\pi\boldsymbol{K}_{n-1}.$$

By theorem (3.5), $\pi \in \boldsymbol{S}_n(\alpha)$ may only exchange d_{α_i} with d_{α_j} if $\lambda_i = \lambda_j$ and so we may choose the coset representative π in $\pi\boldsymbol{E}_n(\alpha)$ to be that element which maps ordered λ_i-tuples into λ_j-tuples ($\lambda_i = \lambda_j$) without changing the internal ordering. These elements π form a group $\boldsymbol{H}$ isomorphic to $\boldsymbol{F}_n(\alpha)$ and hence to $\boldsymbol{M}/\boldsymbol{K}_{n-1}$. The group $\boldsymbol{H}$ will be regarded either as a subgroup of $\boldsymbol{S}_n$ or of $\boldsymbol{S}_r$ according to context.

Define the unitary matrix P by

$$P_{ij} = \prod_{s=1}^{r} \langle\mu_s\rangle(d^{-1}_{\alpha_{\pi(s)}}a_\pi d_{\alpha_s})_{i_s j_s} \tag{6.3}$$

where $\pi \in \boldsymbol{H}$, $d_{\alpha_r} = 1$ and $i = i_1 i_2 \ldots i_r$. Then, using equation (6.2)

$$a_\pi(d_{\alpha_1}\psi^{t_1}_{i_1 w_1}, d_{\alpha_2}\psi^{t_2}_{i_2 w_2}, \ldots, d_{\alpha_r}\psi^{t_r}_{i_r w_r})P^{-1}_{ij} = (d_{\alpha_{\pi(1)}}\psi^{t_1}_{j_1 w_1}, \ldots, d_{\alpha_{\pi(r)}}\psi^{t_r}_{j_r w_r}) \tag{6.4}$$

where the summation convention is being used. For convenience we shall write

$$\langle d_\alpha\psi|^t_{iw} = (d_{\alpha_1}\psi^{t_1}_{i_1 w_1}, \ldots, d_{\alpha_r}\psi^{t_r}_{i_r w_r}).$$

The following results hold by proofs similar to those given in § 5.

Lemma (6.1).

The action of $a_\pi \ldots P^{-1}$ on $\langle d_\alpha\psi|^t_{iw}$ is independent of the chosen coset representative of $\boldsymbol{K}_{n-1}$ in $\boldsymbol{M}$.

Lemma (6.2).

Let $\pi \in \boldsymbol{H}$, then the map $\psi_\alpha: \pi \to a_\pi \ldots P(\pi)^{-1}$ is an isomorphism of $\boldsymbol{H}$ onto a group of operators on the space $V_{(\alpha)}$, where $P(\pi)_{ij} = P_{\pi(i)j}$.

What is different from § 5 is that $a_\pi \ldots P(\pi)^{-1}$ will not in general represent a permutation. However, lemma (5.3) will still hold if some of the double coset representatives are equal, so using equation (6.2), we obtain

$$\begin{aligned} a_\pi\sigma\langle d_\alpha\psi|^t_{iw}P(\pi)^{-1} &= \sigma(d_{\alpha_{\pi(1)}}\psi^{t_1}_{i_{\pi(1)}w_1}, \ldots, d_{\alpha_{\pi(r)}}\psi^{t_r}_{i_{\pi(r)}w_r}) \\ &= \sigma\pi^{-1}(d_{\alpha_1}\psi^{t_{\pi^{-1}(1)}}_{i_1 w_{\pi^{-1}(1)}}, \ldots, d_{\alpha_r}\psi^{t_{\pi^{-1}(r)}}_{i_r w_{\pi^{-1}(r)}}) \end{aligned} \tag{6.5}$$

where $\sigma \in \boldsymbol{S}_n$.

As before, let $V_{\tilde{\tau}(\alpha)}$ be the carrier space for the UR

$$\underbrace{(D_{\alpha_{\tau(1)}} \otimes \ldots \otimes D_{\alpha_{\tau(r)}})}_{n \text{ products}} \downarrow \boldsymbol{K}_{n-1}.$$

Induce the representation of $\boldsymbol{K}_{n-1}$ carried by $\bigoplus_i V_{\widehat{\tau_i^{-1}(\alpha)}}$ up to $\boldsymbol{M}$, where τ_i is defined by equation (5.9), and denote the carrier space of this induced representation by $W(\alpha)$. It can be shown that $W(\alpha)$ is spanned by the functions

$$\{a_\pi \tau_i \langle d_\alpha \psi|^t_{jw} P(\pi)^{-1} : \text{for all } j, w, t, \tau_i, \pi\} = \{\tau_i \langle d_\alpha \psi|^t_{\pi(j)w}\}$$

where

$$\boldsymbol{S}_n = \bigcup_{\pi \in \boldsymbol{H}, i} \tau_i \pi \boldsymbol{E}_n(\alpha).$$

Now the permutations of $\langle d_\alpha \psi|^t_{jw}$ by an element of $\boldsymbol{E}_n(\alpha)$ are already present in $V_{(\alpha)}$. To obtain all the permutations $\pi \langle d_\alpha \psi|^t_{jw}$, where $\pi \in \boldsymbol{H}$, it is sufficient to consider the set $\langle d_\alpha \psi|^t_{\pi(j)w}$ above. This is because, by theorem (3.5), π only interchanges functions $\psi^{t_s}_{j_s w_s}$ belonging to the same tensor space, and the indices t, w of such functions vary over the same set. Hence, $W(\alpha)$ contains all the permutations of the basis elements of $V_{(\alpha)}$.

From the definition of the symmetry class R_{μ_t}, we see that

$$V_{(\alpha)} = \bigoplus_{\mu} R_{\mu_1} \otimes \ldots \otimes R_{\mu_r} \tag{6.6}$$

where the direct sum is taken over all the representations $[\mu_t]$ of $\boldsymbol{S}_{\lambda_t} (t = 1, \ldots, r)$. Define $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ to be the space generated by $\boldsymbol{S}_n$ acting on $R_{\mu_1} \otimes \ldots \otimes R_{\mu_r}$, then

$$W(\alpha) = \bigoplus_{\mu} R_{\mu_1} \odot \ldots \odot R_{\mu_r}. \tag{6.7}$$

Each $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ is invariant under $\boldsymbol{S}_n$ and so the contribution to the character $\chi_\nu(a_\pi)$ in $\boldsymbol{M}$ from the subspace will be zero unless a_π maps it into itself. From equation (6.5) the condition for this is that the spaces generated by $\{\psi^{t_s}_{j_s w_s} : \text{all } j_s\}$ and $\{\psi^{t_{\pi^{-1}(s)}}_{j_s w_{\pi^{-1}(s)}} : \text{all } j_s\}$ must carry the same UR $\langle \mu_{t_s} \rangle$ of $\boldsymbol{K}$ for all $s (s = 1, \ldots, r)$ although they need not be identical spaces if $\dim [\mu_{t_s}] > 1$. We shall now restrict attention to spaces $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ which are closed under operation with $a_\pi \in \boldsymbol{M}$ and for convenience we drop the upper index t_s on the functions $\psi^{t_s}_{j_s w_s}$.

Define the operator $\mathscr{P}^\nu_{\lambda\kappa}$ as follows:

$$\mathscr{P}^\nu_{\lambda\kappa} = \frac{f_\nu}{n!} \sum_{\sigma \in \boldsymbol{S}_n} [\nu](\sigma)^*_{\lambda\kappa} \sigma. \tag{6.8}$$

This operator transforms a vector belonging to the κth row of $[\nu]$ into one belonging to the λth row and annihilates all other vectors. The projection operator onto the space Ω^ν is given by

$$\mathscr{P}^\nu = \sum_\kappa \mathscr{P}^\nu_{\kappa\kappa}. \tag{6.9}$$

If $\psi \in R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ then $\psi = \Sigma_\nu \mathscr{P}^\nu \psi$ is a unique decomposition of ψ into symmetrized parts. Using lemma (5.3)

$$\langle a_\pi \psi, \psi \rangle = \sum_\nu \langle a_\pi \mathscr{P}^\nu \psi, \psi \rangle. \tag{6.10}$$

So given an orthonormal basis $\{\psi_i\}$, the contribution to the character $\chi_\nu(a_\pi)$ from the space $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ is $\Sigma_i \langle a_\pi \mathscr{P}^\nu \psi_i, \psi_i \rangle$.

Take the basis for $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ to be $\{\tau\langle d_\alpha\psi|_{iw} : \boldsymbol{S}_n = \bigcup \tau \boldsymbol{E}_n(\alpha)\}$, then

$$
\begin{aligned}
a_\pi \mathscr{P}^\nu_{\lambda\kappa}\tau\langle d_\alpha\psi|_{iw} &= \mathscr{P}^\nu_{\lambda\kappa}\tau\langle d_\alpha\psi|_{\pi(j)w} P(\pi)_{ji} \\
&= \frac{f_\nu}{n!}\sum_{\sigma\in\boldsymbol{S}_n} [\nu](\sigma)^*_{\lambda\kappa}\sigma\tau\pi^{-1}\langle d_\alpha\psi|_{j\pi^{-1}(w)} P(\pi)_{ji} \\
&= \frac{f_\nu}{n!}\sum_{\sigma\in\boldsymbol{S}_n} [\nu](\tau\sigma\pi\tau^{-1})^*_{\lambda\kappa}\tau\sigma\langle d_\alpha\psi|_{j\pi^{-1}(w)} P(\pi)_{ji}.
\end{aligned}
$$

Using equation (6.2) we obtain (no summation convention)

$$
\begin{aligned}
&\langle a_\pi \mathscr{P}^\nu\tau\langle d_\alpha\psi|_{iw}, \tau\langle d_\alpha\psi|_{iw}\rangle \\
&\quad = \frac{f_\nu}{n!}\sum_{\sigma\in\boldsymbol{E}_n(\alpha)} \chi_{[\nu]}(\sigma\pi)([\mu_1]\otimes\ldots\otimes[\mu_r])(\sigma)_{w\pi^{-1}(w)} P(\pi)_{ii}.
\end{aligned}
$$

Hence the contribution to the character from the space $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ is obtained by summing over τ, i, w:

$$
\chi^\mu_\nu(a_\pi) = \frac{f_\nu}{|\boldsymbol{E}_n(\alpha)|}\sum_{\sigma\in\boldsymbol{E}_n(\alpha)} \chi_{[\nu]}(\sigma\pi)\sum_{i,w} P_{\pi(i)i}([\mu_1]\otimes\ldots\otimes[\mu_r])(\sigma)_{w\pi^{-1}(w)}. \tag{6.11}
$$

Let $\pi = (j_1 \ldots j_{p_1}) \ldots (j_{p_{m-1}+1} \ldots j_r)$ as a product of disjoint cycles, including 1-cycles, where we are regarding $\pi \in \boldsymbol{S}_r$. Then

$$
\begin{aligned}
&\sum_w ([\mu_1]\otimes\ldots\otimes[\mu_r])(\sigma)_{w\pi^{-1}(w)} \\
&\quad = \sum_w \prod_{s=1}^r [\mu_s](\sigma)_{w_s w_{\pi^{-1}(s)}} \\
&\quad = \prod_{t=1}^m \chi_{[\zeta_t]}(\sigma^{p_t - p_{t-1}}),
\end{aligned} \tag{6.12}
$$

where $\zeta = \mu_{j_p}$, $p_m = r$ and $p_0 = 0$. Also

$$
\begin{aligned}
\sum_i P_{\pi(i)i} &= \sum_i \prod_{s=1}^r \langle\mu_s\rangle(d^{-1}_{\alpha_{\pi(s)}} a_\pi d_{\alpha_s})_{i_{\pi(s)} i_s} \\
&= \prod_{t=1}^m \chi_{\langle\zeta_t\rangle}(d^{-1}_{\gamma_t} a_\pi^{p_t - p_{t-1}} d_{\gamma_t}),
\end{aligned} \tag{6.13}
$$

where $\gamma = \alpha_{j_p}$. Hence

$$
\chi^\mu_\nu(a_\pi) = \frac{f_\nu}{|\boldsymbol{E}_n(\alpha)|}\sum_{\sigma\in\boldsymbol{E}_n(\alpha)} \chi_{[\nu]}(\sigma\pi)\prod_{t=1}^m \chi_{[\zeta_t]}(\sigma^{p_t - p_{t-1}})\chi_{\langle\zeta_t\rangle_{\gamma_t}}(a_\pi^{p_t - p_{t-1}}). \tag{6.14}
$$

The characters on symmetry classes of a power of a representation are given in Lyubarskii (1960, chap 4).

The character of the symmetrized representation $\Gamma_{[\nu]}$ of $W(\alpha)$ is

$$
\chi_\nu(a_\pi) = \sum_\mu \chi^\mu_\nu(a_\pi), \tag{6.15}
$$

where $\chi^\mu_\nu(a_\pi)$ is given by equation (6.14) only if a_π maps $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ into itself, otherwise it is zero. When $|\boldsymbol{E}_n(\alpha)| = 1$, then $[\zeta_t] = [1]$ and we obtain equation (5.18). The UR $\Gamma_{[\nu]} \uparrow \boldsymbol{G}$ is carried by the subspace $T(\alpha)^\nu$ and $\boldsymbol{\Omega}^\nu$ defined in equation (5.17).

It is possible to say in advance whether there will be a contribution to the symmetry class $\boldsymbol{\Omega}^\nu$. The space $R_{\mu_1} \otimes \ldots \otimes R_{\mu_r}$ carries the representation $[\mu_1] \otimes \ldots \otimes [\mu_r]$ of $\boldsymbol{E}_n(\alpha)$ $d_\mu = d_{\mu_1} \ldots d_{\mu_r}$ times and hence $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$, which is obtained by operating with the coset representatives of $\boldsymbol{E}_n(\alpha)$ in $\boldsymbol{S}_n$, carries the representation

$$([\mu_1] \otimes \ldots \otimes [\mu_r]) \uparrow \boldsymbol{S}_n = \bigoplus_\nu g_{\mu\nu}[\nu] \tag{6.16}$$

of $\boldsymbol{S}_n$ d_μ times. The left-hand side is known as the outer direct product of the representations $[\mu_1], \ldots, [\mu_r]$ and more details may be found in Hamermesh (1964, chap 7). The right-hand side is a decomposition into representations $[\nu]$ of $\boldsymbol{S}_n$. Hence there is a contribution to the symmetry class $\boldsymbol{\Omega}^\nu$ only if $g_{\mu\nu} \neq 0$.

From the theory of outer direct products, it follows that the identity representation $[n]$ of $\boldsymbol{S}_n$ is contained in the decomposition only if $[\mu_t] = [\lambda_t]$ $(t = 1, \ldots, r)$, and the alternating representation $[1^n]$ is contained in the decomposition only if $[\mu_t] = [1^{\lambda_t}]$ $(t = 1, \ldots, r)$. In both cases the g-factor is one. Also, since there is only one value of w

$$\sum_w ([\mu_1] \otimes \ldots \otimes [\mu_r])(\sigma)_{w\pi^{-1}(w)} = ([\mu_1] \otimes \ldots \otimes [\mu_r])(\sigma). \tag{6.17}$$

Using the Frobenius reciprocity theorem in equation (6.11), we have

$$\chi_n(a_\pi) = \sum_i P_{\pi(i)i} = (-1)^\pi \chi_{1^n}(a_\pi). \tag{6.18}$$

In fact, since $\dim [\lambda_t] = \dim [1^{\lambda_t}] = 1$ $(t = 1, \ldots, r)$, by inspection of equation (6.5) we see that $a_\pi \ldots P(\pi)^{-1}$ does represent a permutation in both of the above cases. Also, $R_{\lambda_1} \odot \ldots \odot R_{\lambda_r}$ is closed under both $\boldsymbol{S}_n$ and $\boldsymbol{M}$, so we may extend the analysis of § 5 to this case. Suppose we have

$$([\lambda_1] \otimes \ldots \otimes [\lambda_r]) \uparrow \bar{\boldsymbol{S}}_n = \bigoplus_\nu g_{\lambda\nu}[\nu]. \tag{6.19}$$

Then $R_{\lambda_1} \odot \ldots \odot R_{\lambda_r}$ carries the representation Γ of $\boldsymbol{M}$, where

$$\Gamma(a_\pi) = \bigoplus_\nu g_{\lambda\nu}[\nu](\bar{\pi}) \otimes P(\pi). \tag{6.20}$$

This can be seen more easily if we consider the elements of $R_{\lambda_1} \odot \ldots \odot R_{\lambda_r}$ which transform according to the ith row of $\langle\lambda_1\rangle_{\alpha_1} \otimes \ldots \otimes \langle\lambda_r\rangle_{\alpha_r}$. Clearly, they form a basis for the representation $([\lambda_1] \otimes \ldots \otimes [\lambda_r]) \uparrow \boldsymbol{S}_n$, so we may take a new set of basis functions $\{\psi_{si}\}$ which, for fixed i, form a basis for the representation $\bigoplus g_{\lambda\nu}[\nu]$, defined by equation (6.19), in fully reduced form. If $\sigma \in \boldsymbol{S}_n$, $l \in \boldsymbol{K}_{n-1}$ we have

$$\bar{\sigma}\psi_{si} = \psi_{ti}(\bigoplus g_{\lambda\nu}[\nu](\bar{\sigma}))_{ts}$$

$$l\psi_{si} = \psi_{sj}(\langle\lambda_1\rangle_{\alpha_1} \otimes \ldots \otimes \langle\lambda_r\rangle_{\alpha_r})(l)_{ji}.$$

Hence

$$a_\pi\psi_{si} = \psi_{tj}P(\pi)_{ji}(\bigoplus g_{\lambda\nu}[\nu](\bar{\pi}))_{ts},$$

as required, so

$$\Gamma_{[\nu]}(a_\pi) = g_{\lambda\nu}[\nu](\bar{\pi}) \otimes P(\pi). \tag{6.21}$$

From equation (6.19) we have

$$([1^{\lambda_1}] \otimes \ldots \otimes [1^{\lambda_r}]) \uparrow \bar{S}_n = \bigoplus g_{\lambda\nu}[\tilde{\nu}], \tag{6.22}$$

where $[\tilde{\nu}] = [\nu] \otimes [1^n]$ is the conjugate representation to $[\nu]$. Hence $R_{1^{\lambda_1}} \odot \ldots \odot R_{1^{\lambda_r}}$ carries the representation Γ of $\boldsymbol{M}$ where

$$\Gamma(a_\pi) = \bigoplus_\nu g_{\lambda\nu}[\tilde{\nu}](\bar{\pi}) \otimes P(\pi), \tag{6.23}$$

and

$$\Gamma_{[\tilde{\nu}]}(a_\pi) = g_{\lambda\nu}[\tilde{\nu}](\bar{\pi}) \otimes P(\pi). \tag{6.24}$$

Another special case which is of interest is when $\boldsymbol{S}_n(\alpha) = \boldsymbol{E}_n(\alpha)$. In this case, $\boldsymbol{M} = \boldsymbol{K}_{n-1}$ and so $T(\alpha)$ carries the representation

$$\frac{|\boldsymbol{S}_n|}{|\boldsymbol{E}_n(\alpha)|}[(\langle\mu_1\rangle_{\alpha_1} \otimes \ldots \otimes \langle\mu_r\rangle_{\alpha_r}) \downarrow \boldsymbol{K}_{n-1}] \uparrow \boldsymbol{G},$$

where we are using the notation introduced at the beginning of this section. From equation (6.11) and since $l \in \boldsymbol{K}_{n-1}$ maps $R_{\mu_1} \odot \ldots \odot R_{\mu_r}$ into itself

$$\chi_\nu^\mu(l) = \frac{f_\nu}{|\boldsymbol{E}_n(\alpha)|} \sum_{\sigma \in \boldsymbol{E}_n(\alpha)} \chi_{[\nu]}(\sigma)\chi_{[\mu_1]}(\sigma) \ldots \chi_{[\mu_r]}(\sigma)\chi_{\tilde{\Delta}}(l). \tag{6.25}$$

Applying the Frobenius reciprocity theorem to (6.16) we have

$$[\nu] \downarrow \boldsymbol{S}_{\lambda_1} \times \ldots \times \boldsymbol{S}_{\lambda_r} = \bigoplus_\mu g_{\mu\nu}[\mu_1] \otimes \ldots \otimes [\mu_r]. \tag{6.26}$$

Hence

$$\chi_\nu^\mu(l) = f_\nu g_{\mu\nu} \chi_{\tilde{\Delta}}(l), \tag{6.27}$$

and the space $T(\alpha)^\nu$ carries the representation

$$\bigoplus_\mu f_\nu g_{\mu\nu}[(\langle\mu_1\rangle_{\alpha_1} \otimes \ldots \otimes \langle\mu_r\rangle_{\alpha_r}) \downarrow \boldsymbol{K}_{n-1}] \uparrow \boldsymbol{G}. \tag{6.28}$$

7. Conclusion

We conclude by giving a brief summary of the method to be used to symmetrize a given induced representation. First, we work out a standard set of double coset representatives as described in § 2. Then pick an n-tuple (α) and reduce all the permutations $\hat{\sigma}(\alpha)$, $\sigma \in \boldsymbol{S}_n$, to standard form. If there are less than $n!$ distinct n-tuples resulting, the group $\boldsymbol{S}_n(\alpha)$ is nontrivial. By (3.5) and theorem (2.5)

$$\boldsymbol{S}_n(\alpha) = \{\sigma \in \boldsymbol{S}_n : \hat{\sigma}(\alpha) \sim (\alpha)\}. \tag{7.1}$$

In this way we split the standard set of double coset representatives into disjoint orbits. The next step is to construct the group $\boldsymbol{M}$ associated with each orbit, as described in theorem (3.7), and use the results of §§ 5 and 6 to obtain the symmetrized representations of $\boldsymbol{M}$. Hence we obtain the symmetrized representations of $\boldsymbol{G}$ as induced representations. The special case $\boldsymbol{S}_n(\alpha) = \boldsymbol{E}_n(\alpha)$ and the simplifications which occur when we are only interested in the totally symmetric and antisymmetric representations are dealt with at the end of § 6.

Clearly the method may be adapted to calculate the nth Kronecker power of $(D \uparrow G)$. We take one representative (α) from each orbit and find the representation

$$[(D_{\alpha_{n-1}} \otimes \ldots \otimes D) \downarrow \boldsymbol{K}_{n-1}] \uparrow \boldsymbol{G}.$$

If the orbit has order t, then this representation appears t times in the decomposition of the nth Kronecker power.

In a subsequent paper, we shall apply our results directly to space group representations, and give a step by step procedure for finding symmetrized cubes. This should be compared with the work of Lewis (1973) where a full group method has been adapted to work out the totally symmetric nth power of a space group representation.

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